THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH3070 Introduction to Topology 2017-2018
Solution of Tutorial Classwork 5

1. Denote the quotient map by $\pi : X \to X/\sim$.

   Note that $\pi^{-1}(\emptyset) = \emptyset \in \mathfrak{T}_X$ and $\pi^{-1}(X/\sim) = X \in \mathfrak{T}_X$. Hence $\emptyset \in \mathfrak{T}_{\text{quot}}$ and $X/\sim \in \mathfrak{T}_{\text{quod}}$.

   Given an index set I and a collection of open sets $\{U_\alpha\}_{\alpha \in I}$ with $\pi^{-1}(U_\alpha) \in \mathfrak{T}_X$, we have $\pi^{-1}(\cup_{\alpha \in I} U_\alpha) = \cup_{\alpha \in I} \pi^{-1}(U_\alpha)$. Since $\mathfrak{T}_X$ is a topology, we have $\cup_{\alpha \in I} \pi^{-1}(U_\alpha) \in \mathfrak{T}_X$. Hence $\cup_{\alpha \in I} U_\alpha \in \mathfrak{T}_{\text{quot}}$.

   Given a collection of open sets $U_1, U_2, \ldots, U_n$ with $\pi^{-1}(U_i) \in \mathfrak{T}_X$, we have $\pi^{-1}(\cap_{i=1}^n U_i) = \cap_{i=1}^n \pi^{-1}(U_i)$. Since $\mathfrak{T}_X$ is a topology, we have $\cap_{i=1}^n \pi^{-1}(U_i) \in \mathfrak{T}_X$. Hence $\cap_{i=1}^n U_i \in \mathfrak{T}_{\text{quot}}$.

   As a result, $\mathfrak{T}_{\text{quot}}$ is a topology.

2. *(Counter example derived from lecture notes)* Let $X = ([-1,1] \times \{0\}) \cup ([-1,1] \times \{1\})$ with subsapce topology induced from $\mathbb{R}^2$. Define the equivalent relation on X by identifying the points $(x,0)$ with $(x,1)$ for all $x \neq 0$. Note that the resulting space $X/\sim$ is not Hausdorff. If we define another equivalent relation by further identifying the points $(0,0)$ and $(0,1)$, then the resulting space is just $[-1,1]$ with standard topology. The resulting space is indeed Hausdorff.

   (Tricky counter example) Consider the two points set $X = \{a, b\}$ with the indiscrete topology $\{\emptyset, \{a,b\}\}$. Clearly X is not Hausdorff. Define an equivalent relation by identifying a and b. Then we have $X/\sim = \{[a]\}$ and $\mathfrak{T}_{\text{quot}} = \{\{[a]\}\}$. Clearly the resulting space is Hausdorff.

3. * Define a function $F : (X/\sim, \mathfrak{T}_{\text{quot}}) \to (\mathbb{R}, \mathfrak{T}_{\text{std}})$ by $F([x,y]) = y - x^2$. Since the value $y - x^2$ is constant on each equivalent classes, the function is well-defined.

   For any $c \in \mathbb{R}$, we have $F([0,c]) = c$. Hence F is surjective. If $F([x_1,y_1]) = F([x_2,y_2])$, then $y_1 - x_1^2 = y_2 - x_2^2$. This implies that $[x_1,y_1] = [x_2,y_2]$. Hence F is injective.

   Furthermore, pick any open set $U \subset \mathbb{R}$, we have $f^{-1}(U) \in \mathfrak{T}_{\text{std}}$. Hence $F^{-1}(U) = \pi(f^{-1}(U))$ is open. This shows that F is continuous.

   Finally, pick any open set $V \subset X/\sim$ and consider $F(V) \subset \mathbb{R}$. Pick a point $[x_0, y_0] = [0, y_0 - x_0^2] \in V$ with $F([0, y_0 - x_0^2]) \in F(V)$. Since $V \subset X/\sim$ is open, $\pi^{-1}(V) \subset \mathbb{R}^2$ is open. In particular, there exists $\epsilon > 0$ such that $(0,t) \in V$ for all t with $|t - (y_0 - x_0^2)| < \epsilon$. This implies that $t \in F(V)$ for all $|t - (y_0 - x_0^2)| < \epsilon$. Hence F maps open set to open set (or in other word, F^{-1} is continuous).

   As a result, F is a homeomorphism.